A HYBRID, VARIATIONAL 3D SMOOTHER FOR ORPHANED SHELL MESHES

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ABSTRACT

In the present paper, a hybrid, variational, user-controlled, 3D mesh smoothing algorithm is proposed for orphaned shell meshes. The smoothing model is based on a variational combination of energy and equi-potential minimization theories. A variety of smoothing techniques for predicting a new location for the node-to-smooth are employed. Each node is moved according to a specific smoothing algorithm so as to keep element included angles, skew and distortion to a minimum. The variational smoother selection logic is based on nodal valency and element connectivity pattern of the node to smooth. Results show its consistency with both quadrilateral and quad-dominant meshes with a significant gain over conventional Laplacian schemes in terms of mesh quality, stability, user control and flexibility.

Keywords: smoothing, mesh, Laplace, incenter, concave, distortion

1. INTRODUCTION

With a very strong legacy of finite element data models, in most engineering industries, more and more legacy FEM data is being read in as building blocks for new designs. This is fast changing the engineering design world to an analysis-driven design as opposed to traditional CAD-based design.

In this light, one of the biggest challenges that the designer faces today is to create geometry from legacy FEM data that is partially altered. The first step towards building surfaces or geometry abstractions from the input mesh is to smooth shell meshes in 3D space. Very few 3D smoothers, known today, can meet this challenge. Such a smoother could also be looked at as a preliminary "mesh morphing" technique.

2. PAST RESEARCH

2.1 Laplacian Schemes

The simplest mesh smoothing technique is Laplacian smoothing where a node is moved to the centroid of its neighboring vertices [14,12 5]. This method operates heuristically, and has no control on mesh quality and often throws nodes outside concave domains producing inverted or invalid elements.

2.2 Laplace Variants

Several variants of the Laplacian smoothing technique include "smart " Laplacian methods [6, 12] where element distortion metrices are checked before the node is moved. Length or area weighted Laplacian methods [12] are able to influence repositioning with initial location. In concave domains, however, these techniques still fail to produce stable meshes. Sometimes, area weighted methods help sense inverted elements and can be leveraged by smart Laplacian approaches to prevent moving such nodes. Haber et al [9] proposed a family of Isoparametric-Laplacian mesh control techniques, meant specially

for transfinite meshes. These schemes tend to produce better skewed quadrilaterals but have most of the other Laplacian disadvantages.

2.3 Optimization Based Methods

In optimization based smoothing, nodes are not moved based on a heuristic algorithm such as Laplacian smoothing, they are moved to minimize a given distortion metric. The distortion metrices are related to the following -

max/min included angles of elements, element skew value, element aspect ratio, element area, element edge length etc.

One of the early optimization techniques was developed by Parthasarathy and Kodiyalam [16]. They solved a non-linear optimization problem in an effort to repair quad-tree and octree meshes. Shephard and Georges [17] reported similar findings. Amenta et al [1] used linear programming techniques to solve

triangular meshes locally. Jacquotte and Coussement [11] developed an optimization based approach for both 2D and 3D structural grids. Freitag et al [6] proposed a local optimization technique for 2D triangular meshes that can serve as the core of an efficient parallel algorithm. Later, Freitag and Oliver-Gooch [7] extended that to 3D grids. These methods produced extremely good quality meshes but involved element repair work like edge-swapping, bad element collapse etc. Moreover the efficiency of these optimization based smoothers are about five to 40 times slower than Laplacian smoothing. Both Freitag [8] and Canann et al [4] later, combined several smart Laplacian methods with optimization-based techniques to create hybrid algorithms to improve efficiency. These methods, however, fail to recognize and preserve mapped meshes.

2.4 Non-Laplacian Methods

Non-Laplacian, physics-based or non-iterative, direct smoothing algorithms have also emerged in the recent past. Lohner et al [15] used a spring mounted system between nodes to smooth. Shimada [18] in his bubble packing model proposed a method where close nodes repel and distant ones attract each other. Bossen and

3. PROBLEM STATEMENT

An extensive study of mesh smoothing algorithms in open literature, proves that there is still no one grid smoothing technique that can work on both 2D and 3D mixed shell meshes to produce good quality elements, give a reasonable control to the user, preserve mapped meshes, not produce inverted elements and yet has an efficiency that compares to Laplacian smoothing.

The present paper, in an effort to fill the gap existing in mesh smoothing literature, proposes a new variational shell mesh smoothing technique in 3D space that symbiotically combines old and some new methods. The variational algorithm smoothes each node according to a specific smoothing technique. The smoothing method selection depends on nodal valency and connectivity pattern of the concerned node and is largely postulated based on the author's experiments on different smoothing techniques on a variety of mesh patterns. The algorithm has the following properties

- It is iterative
- Works on 1D, 2D and 3D space
- Almost as efficient as Laplacian smoothing
- Gives several controls to the user

Heckbert [3] proposed an exponential function with similar properties. These models work in 2D space and are not general purpose. A recent direct, noniterative approach [2] computes artificial stiffness matrices for the mesh to smooth and tries to minimize the strain-energy of the system. While the results are interesting, such methods tend to loose their computational advantage for large models. Tam and Armstrong[19] developed an integer programming based mesh control algorithm for mapped quadrilateral and hexahedral meshes. This method works on a collection of connected subregions and is quite special purpose. Zhou and Shimada [20] have recently proposed an angle smoother in 2D that tends to mount torsion springs between nodes and minimize the system energy. Results prove its merit on certain concave domains and mixed meshes. Winslow smoothing is another efficient technique to reposition nodes of predominantly structured elliptic meshes. Knupp's [21] investigation in this area provides important results. Knupp [22] in another recent paper on mesh smoothing of unstructured quad meshes, proposes "condition numbers" as a good yardstick for measuring mesh quality.

- Does not destroy mapped/structured meshes or mesh regions
- Does not produce inverted elements
- Does not move boundary nodes, or nodes on hard points or nodes with loads
- Improves element included angles, average element skew and hence mesh quality
- Many users handling geometry-associated meshes desire to move nodes off the surface in an effort to reduce element warp and skew. For geometry associated meshes, this algorithm would allow the user to move nodes off the surface within a desired tolerance, if needed.
- Uses different smoothing method for each node in the mesh.
 - 1. An incenter-based approach for triangles
 - Isoparametric-Laplace for general quad only meshes
 - Equipotential (Winslow) smoothing for mapped regions
 - Combined incenter and Laplacian smoothing for free mixed meshes
 - 5. Does a quick region check for each node
 - 6. Does an smart interior angle screening (optional)
 - 7. Constrains the node to move within a specified tolerance (optional)

4. PROPOSED VARIATIONAL SMOOTHING MODEL

The governing equation according to the proposed algorithm, for repositioniong node i connected to N elements, can be written as

$$Pi' = \sum_{n=1}^{N} \operatorname{Fn}(C,V) * \Omega n (C,V) \quad (1)$$

Scheme A: Isoparametric-Laplace smoothing

Conventional Laplacian methods tend to ignore the effect of neighboring nodes that are not directly connected to the node-to-smooth. Furthermore, being heuristic and completely independent of element topology, these techniques do not guarantee low skewness for quad elements. This is a strong requirement for most structural engineering applications. generation Isoparametric mesh fundamentals, derived from the shape functions of quadrilateral isoparametric elements [5], predict location of nodes connected to quadrilateral elements. This technique makes a concious effort to generate grids that adequately represent bounding contours and tend to produce squarish quad elements that have low skew. When this technique is combined with Laplacian grid generation procedures, a more uniform mesh results that have a lower skew value than ordinary Laplacian grids. Such an Isoparametric-Laplacian method is proposed here as a mesh smoothing technique for n-quad connections where n != 2 or 4. A typical n-quad grid is shown in Fig 1.

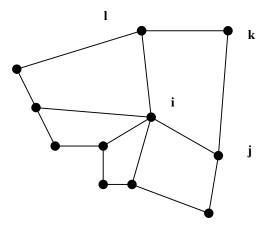


Fig 1: A n-quadrilateral connection (n != 2,4). i-j-k-l denotes an average quad. i is the node to be smoothed.

where Pi' = New position of node i,

Fn = Variational weight factor for n-th element

 Ω n = Positional function for n-th element

C denotes the connectivity pattern of the node, while V indicates its valency

The hybrid smoother uses various repositioning schemes for different types of mesh units. These schemes are disussed below.

From the origin of the isoparametric co-ordinate systems for quadrilateral elements, the new location of node i as shown in Fig I can be expressed as

$$\begin{array}{cccccc} Pi^{'} = & & & & & & \\ & & \sum Wn \; (Pn_{j} \; + Pn_{l} - rPn_{k}) \; \; (\; 2\;) \\ & & N(2-w) & & n = 1 \end{array}$$

where the variational weighting factor and the positional function for each element can be expressed as

$$F_n = W_n$$

 $N(2-r)$ $P_n = (P_{n_j} + P_{n_l} - rP_{n_k})$ (2a)

N is the no of elements connected to node i i-j-k-l represents an average connected quad

r is the coupling factor between Laplacian and isoparametric methods

 Pn_j , Pn_k , Pn_l , represent the position vectors of the j-th, k-th and l-th nodes of the n-the connected quad respectively

Pi' represent the new location of the node to smooth Wn represent weight factors for each connecting element n such that

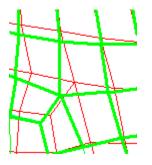
$$\sum_{n=0}^{N} W_n = 1.0;$$

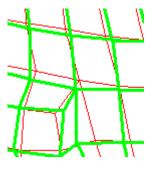
The weight factors can be constructed in numerous possible ways. Some popular examples are length-weighing, area-weighing, weighing elements based on the included angle they make at the node concerned, etc.

Note that when r=0, equation (1) reduces to Laplace smoothing. When r=1.0, a pure isoparametric grid is produced with quad elements showing very low skewness, but the nodal lines of the mesh become zigzag. Experience with scheme has proven that r=0.5 results in good quality meshes with an overall skewness that is almost invariably better than the Laplacian variants.

Figures (2a-2c) demostrate some of the aspects of the isoparametric-laplacian smoothing scheme discussed above. As is evident from figure 2a, the simple Laplacian smoother just a fairly good job smoothing

the 5-valent quad connection but the iso-laplace method produces better quad included angles. The isoparametric smoother, on the contrary, tends to produce a lot more skew elements while trying preserve quad-shapes.





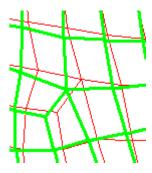


Fig 2a. Laplace

Fig 2b. Isoparametric

Fig 2c. Iso-Laplace

Dark smoothed mesh overlaying a thinner original mesh.

Scheme B: A new Incenter-Based Smoother

This scheme is used for triangles only. Fig 3 shows the a spring-mounted system where the node-to-smooth is assumed to be connected by springs to the incenters of the connected triangles. An energy minimization similar to Laplace smoothing leads to the following expressions.

The new position of node i is obtained as

$$P_{i}^{'}=P_{i}+\sum\limits_{n=1}^{N}W_{n}(P_{n}\text{ - }P_{i})\text{(}3\text{)}\quad\text{where }$$

 $P_i(x, y, z)$ is the position vector of node i $P_n(x, y, z)$ is the incenter vector of element n

N represents the total number of elements connected to node i. The weighing factors W_n account for the initial stiffness of the springs (i.e the effect of initial position of node i). Equation (2) is ambivalent with a length weighted Laplacian method. The weight factors, in this case, can be computed from the length of the incenter vectors as

$$W_{n} = \frac{\parallel V_{n} \parallel}{N \atop N \atop \sum \parallel V_{n} \parallel} \quad (3a)$$

$$n=1$$

where $V_n = P_{n}-P_i$ (3b); V_n is the distance from the concerned node to the incenter of the n-th element.

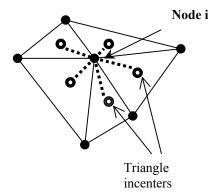


Fig 3. An all-triangle connection. Springs are connected from the node-to-smooth (i) to the Incenters of the connected triangles.

Fig.4a-c illustrate the effectiveness of the proposed algorithm in improving element included angles. Smoothing results are shown for a unit mesh comprising three triangles connected to a single node.

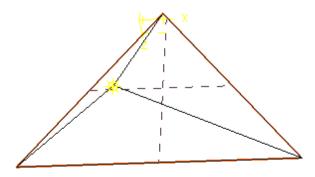


Fig 4a. Three triangles connected to one interior node.

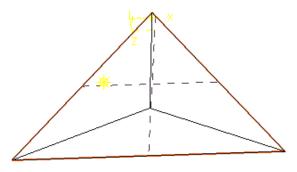


Fig 4b. After Laplace smoothing.

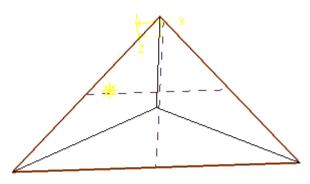


Fig 4c. After incenter smoothing.

The mininum and maximum angles are compared in Table 1.

Table 1. Comparison of element included angles

Smoothing Method	Minimun Element Angle	Maximum Element Angle
Original Mesh	3.7	171.2
Laplacian smoothing	17.3	137.6
Angle Smoothing [20]	16.9	133.2
Proposed variational smoothing	17.4	125.7

Scheme C: Equipotential smoothing

This scheme applies to nodes that are connected to n quads (where n = 4) or has 8 neighboring nodes. Fig.5 depicts a node connected to 4 quadrilateral elements. Neighboring nodes 1,2..8 are shown.

The governing equation for equipotential (Winslow) smoothing can written for node i as

$$\alpha Pi_{\xi\xi} - 2\beta Pi_{\xi\eta} + \gamma Pi_{\eta\eta} = 0;$$
 (4)

where ξ , η are logical variables that are harmonic in nature, while α , β , γ are constant coefficients that depend on the problem.

From Eqn 1 & 4, after some simplification, one obtains an equation similar to eqn. 3 that describes the new location of the node to smooth, where N = 8. The

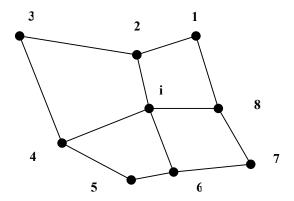


Fig.5: A typical four-quad grid connected to node i

weighing factors of the 8 neighboring nodes are given by

$$W_1 = -\beta/2, W_2 = \alpha, W_3 = \beta/2, W_4 = \gamma, W_5 = -\beta/2, W_6 = \alpha, W_7 = \beta/2, W_8 = \gamma$$
 (4a)

where

$$\begin{split} \alpha &= x_p^2 + y_p^2 + z_p^2 & (4b) \\ \beta &= x_p x_q + y_p y_q + z_p z_q & (4c) \\ \gamma &= x_q^2 + y_q^2 + z_q^2 & (4d) \text{ and} \end{split}$$

$$\begin{array}{lll} x_p = (x_2 - x_6)/2, & y_p = (y_2 - y_6)/2, & z_p = (z_2 - z_6)/2 & (4e) \\ x_q = (x_8 - x_4)/2, & y_q = (y_8 - y_4)/2, & z_q = (z_8 - z_4)/2 & (4f) \end{array}$$

Scheme D: Combined Incenter-IsoLaplacian smoothing for mixed meshes

This scheme applies to nodes that are connected to both quads and triangles and do not satisfy the Winslow criteria. For such connections, a Isoparametric-Laplacian approach is used for quads and the incenter-based smoothing approach is used for triangles. The region check is turned on to prevent mesh distortion in concave domains.

Scheme E: Quad meshes with bivalent nodes

Sometimes we come across a node with a rare valency of 2 (connected to 2 quads). One of these quads is invariably an arrowhead shaped element which is impossible to cure by any smoothing method. For such

bivalent nodes, the node to smooth is deleted and the two connecting quads are merged to form a single quad as shown in Fig.6 a & b.

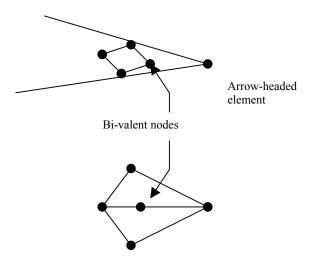


Fig 6a. Elements with bi-valent node

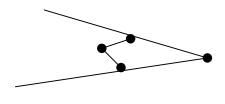


During smoothing, element angles connected to the concerned node are checked to see if they are within the angle limits, or the overall skewness of the connected elements are checked. If all elements connected to the node pass the user-defined criteria, the concerned node is not smoothed. This feature is optional, and when turned on ,the smoother works like a smart Laplacian. Node i, is thus smoothed, only if any included angle of any connected element fails the user-set angular limits at this node. This condition can be mathematically expressed as

Constrained Movement

Since this smoother is most beneficial to users who are trying to modify existing meshes, add features and build geometry from them, many users would prefer the to move nodes in a constrained manner. For example, they may not want their mesh nodes to move

The new location of node
$$i$$
 (P_{i^n}) is given by $P_{i^n} = rP_k + (1-r)P_i$; for $r < 1$ (6a) $P_{i^n} = rP_k$; for $r \ge 1$ (6b) where
$$\frac{dTol}{\parallel P_k - P_i \parallel}$$
 and (6c)



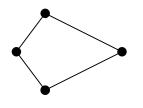


Fig 6b. Bivalent-node removed. Elements merge

$$\begin{split} &\sigma\left(P_{i}\right)\text{ ; if }\theta\text{max} < \alpha_{ji} \parallel_{j=1,\,N} < \theta\text{min} \quad (5)\\ &\text{where}\\ &\sigma\text{ is the smoothing operator}\\ &\sigma\left(P_{i}\right)\text{ indicates, node }i\text{ is smoothed}\\ &N\text{ denotes number of elements connected to node} \end{split}$$

N denotes number of elements connected to node i θ max, θ min are the element allowable angular limits α_{ji} denotes the included angle of element element j at node i.

off their current location by more than a certain amount. Once the user specifies such a node movement tolerance dTol, the nodes are moved from their present location such that they always reside within the sphere of radius R = dTol as shown in Fig.7 a & b.

 $P_i \ , P_{i^*} \ , P_k \ are the node locations before smoothing, after constrained smoothing and after unconstrained smoothing respectively.$

Detail in Fig.7b

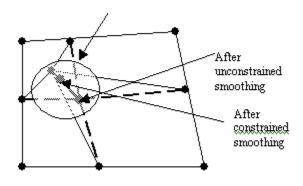


Fig.7a. Constrained smoothing

Region Check

While the hybrid spring system tries to reposition the node so as to relax the worst angles of the connected elements, in certain situations, it is still bound to suffer from the so-called "Laplacian distortion" effect, where, interior nodes might move outside concave boundaries. To ensure that the moved node lies inside the domain of its neighbors, a region, is defined by a bounding box of the neighbors. The node to smooth should not be moved outside this bounding box.

 $X_2 = X_{max}$ of all the initial X's of the centroid

For the node movement to be valid the following conditions need to be met New node location should lie inside bounding box Ω

$$X2 \ge Px \ge X1$$

 $Y2 \ge Py \ge Y1$
 $Z2 \ge Pz \ge Z1$ (7b)

 $(X_1, X_2, Y_1, Y_2, Z_1, Z_2)$

where Px,Py,Pz denote the new coordinates of the node.

Fig. 8 shows a 2D quad-grid where the node to smooth is constrained to move within the bounding box of the three centroids of the three quads.

Convergence

Since most of the methods used in the variational smoothing algorithm are heuristic in nature, the error at each step can be defined by the root mean square of the positional disturbance of the mesh nodes. This can be expressed from equation (1) as $\varepsilon_i = \sum^n (P_i - P_i)^2/n$

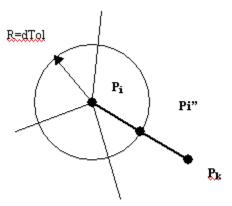


Fig.7b. New node rests on sphere surface

Let the bounding box of the neighbors be expressed as $\Omega(X_1, X_2, Y_1, Y_2, Z_1, Z_2)$ (7a)

where

 $X_1 = X_{min}$ of all the initial X's of the centroid (for quads)/incenters (for triangles) of the neighboring elements

(for quads)/incenters (for triangles) of the neighbors

Y₁, Y₂, Z₁ and Z₂ are expressed identically

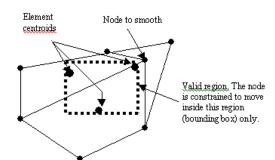


Fig.8: No de movement is constrained to the valid region for this 2D grid.

for the j-th iteration of a mesh that has n nodes. The smoother is assumed to have converged when the following criteria is met ϵ_{j} . $\epsilon_{j-1} < 0.01\epsilon_{1}$.

i = (

The variational logic is controlled by two factors: the valency of the node to smooth, and its element connectivity pattern. Table 2 can best express the rule set adopted.

Table 2: Variational smoothing logic

Valency of node to smooth	No of Quads	No of Trias	Smoothing Method	
2	2		Scheme E: node removed, 2 quads are combined into one	
3	3		SchemeA: Isoparametric-Laplace	
4	4		Scheme C: Equipotential	
n; n > 4	n		Scheme A: Isoparametric-Laplace	
n; n > 2 n!= 8		n	Scheme B: Incenter-based	
8		8	Scheme C: Equipotential	
n; n > 2 n = k + 1	k	1	Scheme D: Incenter-IsoLaplace	

5. RESULTS AND DISCUSSION

Several unitary and large size shell meshes are smoothed using the proposed model. Results are compared with simple Laplace or smart-Laplace smoothing cases or other existing smoothing algorithms. The ability of the smoother to clean

5.1 Ability to preserve mapped-meshes

Most smoothing algorithms do not recognize mapped/structured grids. The proposed angle smoother employs a Winslow smoothing algorithm [12] for nodes that are connected to 4 quadrilaterals or 8 triangles. The benefits of the variational smoother are illustrated in Fig 9a-c.

grossly distorted meshes, maintain stability in concave and triangular regions, preserve mapped or structured meshes, improve interior angles and skewness, and converge efficiently is studied in the following subsections.

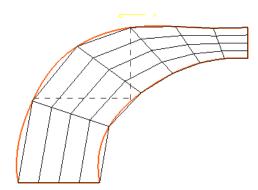


Fig 9a. A 4X6 mapped mesh on a concave region.

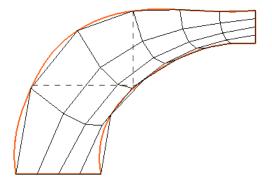


Fig 9b. After Laplace Smoothing. The structured nature of the mesh is lost. Edge elements have almost collapsed.

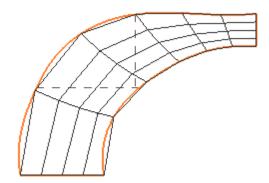


Fig 9c. Result of variational smoothing. All nodes are equipotentially smoothed resulting in a structured or mapped mesh pattern. No element is collapsed.

5.2 Ability to Work On Concave Boundaries

On concave domains Laplacian smoothing often pushes points outside the mesh boundary. The proposed angle smoother prevents that either by employing a Winslow in such regions if the mesh qualifies for it or by a region check. Fig.9a-c show a 5X6 mapped mesh distorted by Laplacian smoothing but completely preserved by the current method.

Distortions caused by Laplacian smoothing in concave domains by throwing nodes outside the mesh region and creating inverting elements can also be prevented by the region check feature of the variational smoother.

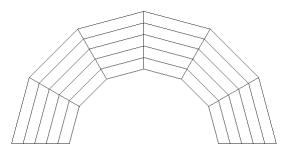


Fig.10a. A 5X6 mapped all quad mesh in a concave region.

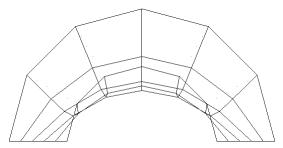


Fig.10b. Result of Laplace smoothing. Nodes move outside the boundary resulting in inverted elements.

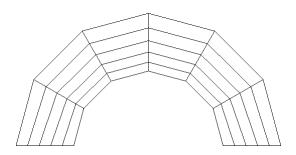


Fig.10c. Result of proposed variational smoothing. No nodes go outside the boundary. The mesh has not been moved around much.

5.3 Ability to Smooth Quad-dominant meshes

Fig.11 shows a portion of a carbody panel that has an initial distorted mesh which is cleaned by the variational smoother as shown in Fig.12. Table 3 compares the performance of the angle smoother to smart-Laplacian smoothing. Once again , it is observed that the number of elements failing included angle check for the angle smoother is half of that produced by smart-Laplacian smoothing.

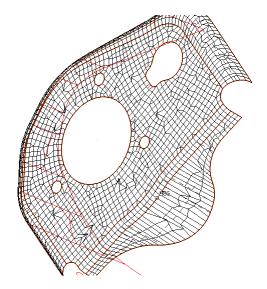


Fig.11: Portion of a carbody panel. Quad dominant mesh, partially mapped mesh with high-skew elements.

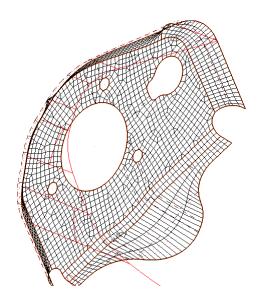


Fig.12: Mesh smoothed by the new variational smoother. All elements are smoothed. Resultant mesh shows mapped grids have been preserved, most high skew elements cured.

Table 3. Comparison of mesh quality for a carbody panel

Smoothing Method	No. of elm failing warp check	No. of elem failing minimum Jacobian check	_	No. of elem failing included angle check
Original mesh	37	89	241	156
After Laplace smoothing	5	2	89	31
After variational smoothing	1	2	63	15

6. CONCLUSION

It has been clearly proven by several past researches, that Laplacian smoothing or its variants have certain inherent shortcomings that can not be cured by a single alternate smoothing algorithm. More over, most schemes, are restricted to either two-dimensional or monolithic grids. The proposed variational smoother uses a hybrid smoothing mechanism where different smoothing (both conventional and

innovative) algorithms are used for each node depending on its connectivity. For all triangular meshes, an incenter-based method is proposed that performs better and less destructive than the Laplacian method. For all quadrilateral meshes, an Isoparametric-Laplace method is used that leads to faster convergence and lower skewness. Winslow smoothing is employed for mapped-grid units that

preserves structured grids and is non-destructive in concave domains. The method is made more flexible by adding an user control on the element check criteria (included angle or skew) and is further reinforced by a quick region check that often ensures valid meshes in skew and concave contours. Several results exemplify the performance of the proposed variational smoother in comparison to a conventional Laplacian, smart-Laplacian and other schemes.

7. ACKNOWLEDGEMENT

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